

Yang-Baxter basis of Hecke algebra and Casselman's problem (extended abstract)

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Abstract

We generalize the definition of Yang-Baxter basis of type A Hecke algebra introduced by A.Lascoux, B.Leclerc and J.Y.Thibon (Letters in Math. Phys., 40 (1997), 75–90) to all the Lie types and prove their duality. As an application we give a solution to Casselman's problem on Iwahori fixed vectors of principal series representation of p -adic groups.

1 Introduction

Yang-Baxter basis of Hecke algebra of type A was defined in the paper of Lascoux-Leclerc-Thibon [LLT]. There is also a modified version in [Las]. First we generalize the latter version to all the Lie types. Then we will solve the Casselman's problem on the basis of Iwahori fixed vectors using Yang-Baxter basis and Demazure-Lusztig type operator. This paper is an extended abstract and the detailed proofs will appear in [NN].

2 Generic Hecke algebra

2.1 Root system, Weyl group and generic Hecke algebra

Let $\mathcal{R} = (\Lambda, \Lambda^*, R, R^*)$ be a (reduced) semisimple root data cf. [Dem]. More precisely $\Lambda \simeq \mathbb{Z}^r$ is a weight lattice with $\text{rank } \Lambda = r$. There is a pairing $\langle \cdot, \cdot \rangle: \Lambda^* \times \Lambda \rightarrow \mathbb{Z}$. $R \subset \Lambda$ is a root system with simple roots $\{\alpha_i\}_{1 \leq i \leq r}$ and positive roots R^+ . $R^* \subset \Lambda^*$ is the set of coroots, and there is a bijection $R \rightarrow R^*$, $\alpha \mapsto \alpha^*$. We also denote the coroot $\alpha^* = h_\alpha$. The Weyl group W of \mathcal{R} is generated by simple reflections $S = \{s_i\}_{1 \leq i \leq r}$. The action of W on Λ is given by $s_i(\lambda) = \lambda - \langle \alpha_i^*, \lambda \rangle \alpha_i$ for $\lambda \in \Lambda$. We define generic Hecke algebra $H_{t_1, t_2}(W)$ over $\mathbb{Z}[t_1, t_2]$ with two parameters t_1, t_2 as follows. Generators are $h_i = h_{s_i}$, with relations $(h_i - t_1)(h_i - t_2) = 0$ for $1 \leq i \leq r$ and the braid relations $\underbrace{h_i h_j \cdots}_{m_{i,j}} = \underbrace{h_j h_i \cdots}_{m_{i,j}}$, where $m_{i,j}$ is the order of $s_i s_j$ for $1 \leq i < j \leq r$.

We need to extend the coefficients to the quotient field of the group algebra $\mathbb{Z}[\Lambda]$. An element of $\mathbb{Z}[\Lambda]$ is denoted as $\sum_{\lambda \in \Lambda} c_\lambda e^\lambda$. The Weyl group acts on $\mathbb{Z}[\Lambda]$

by $w(e^\lambda) = e^{w\lambda}$. We extend the coefficient ring $\mathbb{Z}[t_1, t_2]$ of $H_{t_1, t_2}(W)$ to

$$Q_{t_1, t_2}(\Lambda) := \mathbb{Z}[t_1, t_2] \otimes Q(\mathbb{Z}[\Lambda])$$

where $Q(\mathbb{Z}[\Lambda])$ is the quotient field of $\mathbb{Z}[\Lambda]$.

$$H_{t_1, t_2}^{Q(\Lambda)}(W) := Q_{t_1, t_2}(\Lambda) \otimes_{\mathbb{Z}[t_1, t_2]} H_{t_1, t_2}(W).$$

For $w \in W$, an expression of $w = s_{i_1} s_{i_2} \cdots s_{i_\ell}$ with minimal number of generators $s_{i_k} \in S$ is called a reduced expression in which case we write $\ell(w) = \ell$ and call it the length of w . Then $h_w = h_{i_1} h_{i_2} \cdots h_{i_\ell}$ is well defined and $\{h_w\}_{w \in W}$ forms a $Q_{t_1, t_2}(\Lambda)$ -basis of $H_{t_1, t_2}^{Q(\Lambda)}(W)$.

2.2 Yang-Baxter basis and its properties

Yang-Baxter basis was introduced in the paper [LLT] to investigate the relation with Schubert calculus. There is also a variant in [Las] for type A case. We generalize that results to all Lie types.

For $\lambda \in \Lambda$, we define $E(\lambda) = e^{-\lambda} - 1$. Then $E(\lambda + \nu) = E(\lambda) + E(\nu) + E(\lambda)E(\nu)$. In particular, if $\lambda \neq 0$, $\frac{1}{E(\lambda)} + \frac{1}{E(-\lambda)} = -1$.

Proposition 1. *For $\lambda \in \Lambda$, if $\lambda \neq 0$, let $h_i(\lambda) := h_i + \frac{t_1 + t_2}{E(\lambda)}$. Then these satisfy the **Yang-Baxter relations**, i.e. if we write $[p, q] := p\lambda + q\nu$ for fixed $\lambda, \nu \in \Lambda$, the following equations hold. We assume all appearance of $[p, q]$ is nonzero.*

$$\begin{aligned} h_i([1, 0])h_j([0, 1]) &= h_j([0, 1])h_i([1, 0]) && \text{if } m_{i,j} = 2 \\ h_i([1, 0])h_j([1, 1])h_i([0, 1]) &= h_j([0, 1])h_i([1, 1])h_j([1, 0]) && \text{if } m_{i,j} = 3 \\ h_i([1, 0])h_j([1, 1])h_i([1, 2])h_j([0, 1]) &= h_j([0, 1])h_i([1, 2])h_j([1, 1])h_i([1, 0]) && \text{if } m_{i,j} = 4 \\ h_i([1, 0])h_j([1, 1])h_i([2, 3]) &= h_j([0, 1])h_i([1, 3])h_j([1, 2]) \\ \times h_j([1, 2])h_i([1, 3])h_j([0, 1]) &= \times h_i([2, 3])h_j([1, 1])h_i([1, 0]) && \text{if } m_{i,j} = 6 \end{aligned}$$

Proof. We can prove these equations by direct calculations. \square

Remark 1. In [Che] I. Cherednik treated Yang-Baxter relation in more general setting. There is also a related work [Kat] by S. Kato and the proof of Theorem 2.4 in [Kat] suggests a uniform way to prove Yang-Baxter relations without direct calculations.

We use the Bruhat order $x \leq y$ on elements $x, y \in W$ (cf.[Hum]). Following [Las] we define the Yang-Baxter basis Y_w for $w \in W$ recursively as follows.

$$Y_e := 1, Y_w := Y_{w'}(h_i + \frac{t_1 + t_2}{w'E(\alpha_i)}) \text{ if } w = w's_i > w'.$$

Using the Yang-Baxter relation above it is easy to see that Y_w does not depend on a reduced expression of w . As the leading term of Y_w with respect to the Bruhat order is h_w , they also form a $Q_{t_1, t_2}(\Lambda)$ -basis $\{Y_w\}_{w \in W}$ of $H_{t_1, t_2}^{Q(\Lambda)}(W)$.

We are interested in the transition coefficients $p(w, v)$ and $\tilde{p}(w, v) \in Q_{t_1, t_2}(\Lambda)$ between the two basis $\{Y_w\}_{w \in W}$ and $\{h_w\}_{w \in W}$, i.e.

$$Y_v = \sum_{w \leq v} p(w, v) h_w, \text{ and } h_v = \sum_{w \leq v} \tilde{p}(w, v) Y_w.$$

Take a reduced expression of v e.g. $v = s_{i_1} \cdots s_{i_\ell}$ where $\ell = \ell(v)$ is the length of v (cf. [Hum]). Then Y_v is expressed as follows.

$$Y_v = \prod_{j=1}^{\ell} \left(h_{i_j} + \frac{t_1 + t_2}{E(\beta_j)} \right)$$

where $\beta_j := s_{i_1} \cdots s_{i_{j-1}}(\alpha_{i_j})$ for $j = 1, \dots, \ell$. The set $R(v) := \{\beta_1, \dots, \beta_\ell\} \subset R^+$ is independent of the reduced expression of v . The Yang-Baxter basis defined in [LLT] is normalized as follows.

$$Y_v^{LLT} := \left(\prod_{j=1}^{\ell} \frac{E(\beta_j)}{t_1 + t_2} \right) Y_v = \prod_{j=1}^{\ell} \left(\frac{E(\beta_j)}{t_1 + t_2} h_{i_j} + 1 \right).$$

Remark 2. *The relation to K -theory Schubert calculus is as follows. If we set $t_1 = 0, t_2 = -1$ and replacing α_i by $-\alpha_i$. Then the coefficient of h_w in Y_v^{LLT} is the localization $\psi^w(v)$ at v of the equivariant K -theory Schubert class ψ^w (cf. [LSS]).*

Let w_0 be the longest element in W . Define $Q_{t_1, t_2}(\Lambda)$ -algebra homomorphism $\Omega : H_{t_1, t_2}^{Q(\Lambda)} \rightarrow H_{t_1, t_2}^{Q(\Lambda)}$ by $\Omega(h_w) = h_{w_0 w w_0}$. Let \star be the ring homomorphism on $\mathbb{Z}[\Lambda]$ induced by $\star(e^\lambda) = e^{-\lambda}$ and extend to $Q_{t_1, t_2}(\Lambda)$.

Proposition 2. (Lascoux [Las] Lemma 1.8.1 for type A case) For $v \in W$,

$$\Omega(Y_{w_0 v w_0}) = \star[w_0(Y_v)]$$

where W acts only on the coefficients.

Proof. When $\ell(v) > 0$ there exists $s \in S$ such that $v = v's > v'$. Using the induction assumption on v' , we get the formula for v . \square

Taking the coefficient of h_w in the above equation, we get

Corollary 1.

$$p(w_0 w w_0, w_0 v w_0) = \star[w_0 p(w, v)].$$

2.3 Inner product and orthogonality

Define inner product $(\ , \)^H$ on $H_{t_1, t_2}^{Q(\Lambda)}(W)$ by $(f, g)^H :=$ the coefficient of h_{w_0} in $f g^\vee$, where $g^\vee = \sum c_w h_{w^{-1}}$ if $g = \sum c_w h_w$. It is easy to see that $(f h_s, g)^H = (f, g h_s)^H$ for $f, g \in H_{t_1, t_2}^{Q(\Lambda)}(W)$ and $s \in S$. There is an involution $\hat{\cdot} : H_{t_1, t_2}^{Q(\Lambda)} \rightarrow$

$H_{t_1, t_2}^{Q(\Lambda)}$ defined by $\hat{h}_i = h_i - (t_1 + t_2)$, $\hat{t}_1 = -t_2$, $\hat{t}_2 = -t_1$. It is easy to see that $\hat{h}_s h_s = -t_1 t_2$ for $s \in S$.

The following proposition is due to A.Lascoux for the type A case [Las] P.33.

Proposition 3. *For all $v, w \in W$,*

$$(h_v, \hat{h}_{w_0 w})^H = \delta_{v, w}.$$

Proof. We can use induction on the length $\ell(v)$ of v to prove the equation. \square

We have another orthogonality between Y_v and $w_0(Y_{w_0 w})$.

Proposition 4. *(Type A case was due to [LLT] Theorem 5.1, [Las] Theorem 1.8.4.)*

For all $v, w \in W$,

$$(Y_v, w_0(Y_{w_0 w}))^H = \delta_{v, w}.$$

Proof. We use induction on $\ell(v)$ and use the fact that if $s \in S$ and $u \in W$, then $Y_u h_s = aY_{us} + bY_s$ for some $a, b \in Q_{t_1, t_2}(\Lambda)$. \square

2.4 Duality between the transition coefficients

Recall that we have two transition coefficients $p(w, v), \tilde{p}(w, v) \in Q_{t_1, t_2}(\Lambda)$ defined by the following expansions.

$$Y_v = \sum_{w \leq v} p(w, v) h_w$$

$$h_v = \sum_{w \leq v} \tilde{p}(w, v) Y_w$$

Below gives a relation between them.

Theorem 1. *(Lascoux [Las] Corollary 1.8.5 for type A case) For $w, v \in W$,*

$$\tilde{p}(w, v) = (-1)^{\ell(v) - \ell(w)} p(vw_0, ww_0).$$

Proof. We will calculate $(h_v, w_0(Y_{w_0 w}))^H$ in two ways. As $h_v = \sum_{w \leq v} \tilde{p}(w, v) Y_w$,

$$(h_v, w_0(Y_{w_0 w}))^H = \tilde{p}(w, v)$$

by the orthogonality on Y_v (Proposition 4). On the other hand, as $h_i + \frac{t_1 + t_2}{E(\beta)} = \hat{h}_i - \frac{t_1 + t_2}{E(-\beta)}$ for $\beta \in R$, we can expand Y_v in terms of \hat{h}_w as follows.

$$Y_v = \sum_{w \leq v} (-1)^{\ell(v) - \ell(w)} \star [p(w, v)] \hat{h}_w.$$

So we have

$$w_0(Y_{w_0 w}) = \sum_{w_0 v \leq w_0 w} (-1)^{\ell(v) - \ell(w)} w_0[\star p(w_0 v, w_0 w)] \hat{h}_{w_0 v}.$$

Then using the orthogonality on h_v (Proposition 3) and Corollary 1,

$$(h_v, w_0(Y_{w_0 w}))^H = (-1)^{\ell(v) - \ell(w)} w_0[\star p(w_0 v, w_0 w)] = (-1)^{\ell(v) - \ell(w)} p(vw_0, ww_0).$$

The theorem is proved. \square

2.5 Recurrence relations

Here we give some recurrence relations on $p(w, v)$ and $\tilde{p}(w, v)$.

Proposition 5. (left p) For $w \in W$ and $s \in S$, if $sv > v$ then

$$p(w, sv) = \begin{cases} \frac{t_1+t_2}{E(\alpha_s)} s[p(w, v)] - t_1 t_2 s[p(sw, v)] & \text{if } sw > w \\ (t_1 + t_2)(\frac{1}{E(\alpha_s)} + 1) s[p(w, v)] + s[p(sw, v)] & \text{if } sw < w. \end{cases}$$

Proof. By the definition we have $Y_{sv} = Y_s s[Y_v]$ from which we can deduce the recurrence formula. \square

We note that by this recurrence we can identify $p(w, v)$ as a coefficient of transition between two bases of the space of Iwahori fixed vectors cf. Theorem 3 below.

Proposition 6. (right p) For $w \in W$ and $s \in S$, if $vs > v$ then

$$p(w, vs) = \begin{cases} \frac{t_1+t_2}{vE(\alpha_s)} p(w, v) - t_1 t_2 p(ws, v) & \text{if } ws > w \\ (t_1 + t_2)(\frac{1}{vE(\alpha_s)} + 1) p(w, v) + p(ws, v) & \text{if } ws < w. \end{cases}$$

Proof. We can use the equation $Y_{vs} = Y_v v[Y_s]$ and taking the coefficient of h_w , we get the formula. \square

Proposition 7. (left \tilde{p}) For $w \in W$ and $s \in S$, if $sv > v$ then

$$\tilde{p}(w, sv) = \begin{cases} -\frac{t_1+t_2}{E(\alpha_s)} \tilde{p}(w, v) + (1 + \frac{t_1+t_2}{E(\alpha_s)})(1 + \frac{t_1+t_2}{E(-\alpha_s)}) s[\tilde{p}(sw, v)] & \text{if } sw > w \\ -\frac{t_1+t_2}{E(\alpha_s)} \tilde{p}(w, v) + s[\tilde{p}(sw, v)] & \text{if } sw < w. \end{cases}$$

Proof. We can prove the recurrence relation using Corollary 2 below. \square

Proposition 8. (right \tilde{p}) For $w \in W$ and $s \in S$, if $vs > v$ then

$$\tilde{p}(w, vs) = \begin{cases} -\frac{t_1+t_2}{wE(\alpha_s)} \tilde{p}(w, v) + (1 + \frac{t_1+t_2}{wE(\alpha_s)})(1 + \frac{t_1+t_2}{wE(-\alpha_s)}) \tilde{p}(ws, v) & \text{if } ws > w \\ -\frac{t_1+t_2}{wE(\alpha_s)} \tilde{p}(w, v) + \tilde{p}(ws, v) & \text{if } ws < w. \end{cases}$$

Proof. We can prove the recurrence relation using Corollary 2 below. \square

3 Kostant-Kumar's twisted group algebra

Let $Q_{t_1, t_2}^{KK}(W) := Q_{t_1, t_2}(\Lambda) \# \mathbb{Z}[W]$ be the (generic) twisted group algebra of Kostant-Kumar. Its element is of the form $\sum_{w \in W} f_w \delta_w$ for $f_w \in Q_{t_1, t_2}(\Lambda)$ and the product is defined by

$$\left(\sum_{w \in W} f_w \delta_w \right) \left(\sum_{u \in W} g_u \delta_u \right) = \sum_{w, u \in W} f_w w(g_u) \delta_{wu}.$$

Define $y_i \in Q_{t_1, t_2}^{KK}(W)$ ($i = 1, \dots, r$) by

$$y_i := A_i \delta_i + B_i \quad \text{where} \quad A_i := \frac{t_1 + t_2 e^{-\alpha_i}}{1 - e^{\alpha_i}}, B_i := \frac{t_1 + t_2}{1 - e^{-\alpha_i}}.$$

Proposition 9. *We have the following equations.*

- (1) $(y_i - t_1)(y_i - t_2) = 0$ for $i = 1, \dots, r$.
- (2) $\underbrace{y_i y_j \cdots}_{m_{i,j}} = \underbrace{y_j y_i \cdots}_{m_{i,j}}$, where $m_{i,j}$ is the order of $s_i s_j$.

Proof. These equations can be shown by direct calculations. \square

By this proposition we can define $y_w := y_{i_1} \cdots y_{i_\ell}$ for a reduced expression $w = s_{i_1} \cdots s_{i_\ell}$. These $\{y_w\}_{w \in W}$ become a $Q_{t_1, t_2}(\Lambda)$ -basis of $Q_{t_1, t_2}^{KK}(W)$.

Remark 3. *This operator y_i can be seen as a generic Demazure-Lusztig operator. When $t_1 = -1, t_2 = q$, it becomes $y_{s_i}^q$ in Kumar's book [Kum](12.2.E(9)). We can also set A_i which satisfies*

$$A_i A_{-i} = \frac{(t_1 + t_2 e^{-\alpha_i})(t_1 + t_2 e^{\alpha_i})}{(1 - e^{\alpha_i})(1 - e^{-\alpha_i})}.$$

For example, if we set $A_i = \frac{t_1 + t_2 e^{\alpha_i}}{1 - e^{\alpha_i}}$ and $t_1 = q, t_2 = -1$ and replace α_i by $-\alpha_i$, it becomes Lusztig's T_{s_i} [Lu1]. If we set $A_i = -\frac{t_1 + t_2 e^{\alpha_i}}{1 - e^{-\alpha_i}}$ and $t_1 = -1, t_2 = v$ and replace α_i by $-\alpha_i$, it becomes \mathcal{T}_i in [BBL].

We can define a $Q_{t_1, t_2}(\Lambda)$ -module isomorphism $\Phi : Q_{t_1, t_2}^{KK}(W) \rightarrow H_{t_1, t_2}^{Q(\Lambda)}(W)$ by $\Phi(y_w) = h_w$. Let $\Delta_{s_i} := A_i \delta_i$. Define $A(w) := \prod_{\beta \in R(w)} \frac{t_1 + t_2 e^{-\beta}}{1 - e^{\beta}}$ and $\Delta_w := A(w) \delta_w$. Then it becomes that $\Delta_{s_{i_1}} \cdots \Delta_{s_{i_\ell}} = A(w) \delta_w = \Delta_w$. In particular, Δ_{s_i} 's satisfy the braid relations. We can show below by induction on length $\ell(w)$.

Theorem 2. *For $w \in W$, we have*

$$\Phi(\Delta_w) = Y_w.$$

Proof. If $w = s_i$, $\Delta_{s_i} = A_i \delta_i = y_i - B_i$. Therefore $\Phi(\Delta_{s_i}) = h_i - B_i = h_i + \frac{t_1+t_2}{E(\alpha_i)} = Y_{s_i}$. If $s_i w > w$, by induction hypothesis we can assume $\Phi(\Delta_w) = Y_w = \sum_{u \leq w} p(u, w) h_u$. As Φ is a $Q_{t_1, t_2}(\Lambda)$ -isomorphism, it follows that $\Delta_w = \sum_{u \leq w} p(u, w) y_u$. Then $\Delta_{s_i w} = \Delta_{s_i} \Delta_w = A_i \delta_i \sum_{u \leq w} p(u, w) y_u = \sum_{u \leq w} s_i [p(u, w)] A_i \delta_i y_u = \sum_{u \leq w} s_i [p(u, w)] (y_i - B_i) y_u = \sum_{u \leq s_i w} p(u, s_i w) y_u$. We used the recurrence relation (Proposition 5) for the last equality. Therefore $\Phi(\Delta_{s_i w}) = \sum_{u \leq s_i w} p(u, s_i w) h_u = Y_{s_i w}$. The theorem is proved. \square

Corollary 2. (*Explicit formula for $\tilde{p}(w, v)$*)

Let $v = s_{i_1} \cdots s_{i_\ell}$ be a reduced expression. Then we have

$$\tilde{p}(w, v) = \frac{1}{A(w)} \sum_{\epsilon = (\epsilon_1, \dots, \epsilon_\ell) \in \{0, 1\}^\ell, s_{i_1}^{\epsilon_1} \cdots s_{i_\ell}^{\epsilon_\ell} = w} \prod_{j=1}^{\ell} C_j(\epsilon)$$

where for $\epsilon = (\epsilon_1, \dots, \epsilon_\ell) \in \{0, 1\}^\ell$, $C_j(\epsilon) := s_{i_1}^{\epsilon_1} s_{i_2}^{\epsilon_2} \cdots s_{i_{j-1}}^{\epsilon_{j-1}} (\delta_{\epsilon_j, 1} A_{i_j} + \delta_{\epsilon_j, 0} B_{i_j})$.

Proof. Taking the inverse image of the map Φ , the equality $h_v = \sum_{w \leq v} \tilde{p}(w, v) Y_w$ becomes

$$y_v = \sum_{w \leq v} \tilde{p}(w, v) \Delta_w = \sum_{w \leq v} \tilde{p}(w, v) A(w) \delta_w.$$

As $v = s_{i_1} \cdots s_{i_\ell}$ is a reduced expression, $y_v = y_{s_{i_1}} \cdots y_{s_{i_\ell}} = (A_{i_1} \delta_{i_1} + B_{i_1} \delta_e) \cdots (A_{i_\ell} \delta_{i_\ell} + B_{i_\ell} \delta_e)$. By expanding this we get the formula. \square

Remark 4. Using Theorem 1, we also have a closed form for $p(w, v)$. We have another conjectural formula for $p(w, v)$ using λ -chain cf. [Nar].

Example 1. Type A_2 . We use notation $A_{-1} = \star(A_1)$, $B_{-1} = \star(B_1)$, $B_{12} = \frac{t_1+t_2}{1-e^{-(\alpha_1+\alpha_2)}}$.

When $v = s_1 s_2 s_1$, $w = s_1$, then $\epsilon = (1, 0, 0), (0, 0, 1)$ and

$$\tilde{p}(s_1, s_1 s_2 s_1) = (A_1 B_{12} B_{-1} + B_1 B_2 A_1) / A_1 = B_{12} B_{-1} + B_1 B_2 = B_2 B_{12}.$$

When $v = s_1 s_2 s_1$, $w = s_2$, then $\epsilon = (0, 1, 0)$ and

$$\tilde{p}(s_2, s_1 s_2 s_1) = (B_1 A_2 B_{12}) / A_2 = B_1 B_{12}.$$

When $v = s_1 s_2 s_1$, $w = e$, then $\epsilon = (0, 0, 0), (1, 0, 1)$ and

$$\tilde{p}(e, s_1 s_2 s_1) = B_1 B_2 B_1 + A_1 B_{12} A_{-1}.$$

4 Casselman's problem

In his paper [Cas] B. Casselman gave a problem concerning transition coefficients between two bases in the space of Iwahori fixed vectors of a principal series representation of a p -adic group. We relate the problem with the Yang-Baxter basis and give an answer to the problem.

4.1 Principal series representations of p -adic group and Iwahori fixed vector

We follow the notations of M.Reeder [Re1, Re2]. Let G be a connected reductive p -adic group over a non-archimedean local field F . For simplicity we restrict to the case of split semisimple G . Associated to F , there is the ring of integers \mathcal{O} , the prime ideal \mathfrak{p} with a generator ϖ , and the residue field with $q = |\mathcal{O}/\mathfrak{p}|$ elements. Let P be a minimal parabolic subgroup (Borel) of G , and A be the maximal split torus of P so that $A \simeq (F^*)^r$ where r is the rank of G . For an unramified quasi-character τ of A , i.e. a group homomorphism $\tau : A \rightarrow \mathbb{C}^*$ which is trivial on $A_0 = A \cap K$, where $K = G(\mathcal{O})$ is a maximal compact subgroup of G . Let $T = \mathbb{C}^* \otimes X^*(A)$ be the complex torus dual to A , where $X^*(A)$ is the group of rational characters on A , i.e. $X^*(A) = \{\lambda : A \rightarrow F^*, \text{ algebraic group homomorphism}\}$. We have a pairing $\langle, \rangle : A/A_0 \times T \rightarrow \mathbb{C}^*$ given by $\langle a, z \otimes \lambda \rangle = z^{\text{val}(\lambda(a))}$. This gives an identification $T \simeq X^{nr}(A)$ of T with the set of unramified quasi-characters on A (cf. [Bum] Exercise 18,19).

Let $\Delta \subset X^*(A)$ be the set of roots of A in G , Δ^+ be the set of positive roots corresponding to P and $\Sigma \subset \Delta^+$ be the set of simple roots. For a root $\alpha \in \Delta$, we define $e_\alpha \in X^*(T)$ by

$$e_\alpha(\tau) = \langle h_\alpha(\varpi), \tau \rangle$$

for $\tau \in T$ where $h_\alpha : F^* \rightarrow A$ is the one parameter subgroup (coroot) corresponding to α .

Remark 5. *As the definition shows, e_α is defined using the coroot $\alpha^* = h_\alpha$. So it should be parametrized by α^* , but for convenience we follow the notation of [Re1]. Later we will identify $e_\alpha(\alpha \in \Delta = R^*)$ with $e^\alpha(\alpha \in R = \Delta^*)$ by the map $*$: $\Delta \rightarrow R$ of root data.*

W acts on right of $X^{nr}(A)$ so that $\tau^w(a) = \tau(waw^{-1})$ for $a \in A$, $\tau \in T$ and $w \in W$. The action of W on $X^*(T)$ is given by $(we_\alpha)(\tau) = e_{w\alpha}(\tau) = e_\alpha(\tau^w)$ for $\alpha \in \Delta$, $\tau \in T$ and $w \in W$.

The principal series representation $I(\tau)$ of G associated to a unramified quasicharacter τ of A is defined as follows. As a vector space over \mathbb{C} it consists of locally constant functions on G with values in \mathbb{C} which satisfy the left relative invariance properties with respect to P where τ is extended to P with trivial value on the unipotent radical N of $P = AN$.

$$I(\tau) := \text{Ind}_P^G(\tau) = \{f : G \rightarrow \mathbb{C} \text{ loc. const. function} \mid f(pg) = \tau\delta^{1/2}(p)f(g) \text{ for } \forall p \in P, \forall g \in G\}.$$

Here δ is the modulus of P . The action of G on $I(\tau)$ is defined by right translation, i.e. for $g \in G$ and $f \in I(\tau)$, $(\pi(g)f)(x) = f(xg)$.

Let B be the Iwahori subgroup which is the inverse image $\pi^{-1}(P(\mathbb{F}_q))$ of the Borel subgroup $P(\mathbb{F}_q)$ of $G(\mathbb{F}_q)$ by the projection $\pi : G(\mathcal{O}) \rightarrow G(\mathbb{F}_q)$. Then we define $I(\tau)^B$ to be the space of Iwahori fixed vectors in $I(\tau)$, i.e.

$$I(\tau)^B := \{f \in I(\tau) \mid f(gb) = f(g) \text{ for } \forall b \in B, \forall g \in G\}.$$

This space has a natural basis $\{\varphi_w^\tau\}_{w \in W}$. $\varphi_w^\tau \in I(\tau)^B$ is supported on PwB and satisfies

$$\varphi_w^\tau(pwb) = \tau\delta^{1/2}(p) \text{ for } \forall p \in P, \forall b \in B.$$

4.2 Intertwiner and Casselman's basis

From now on we always assume that τ is regular i.e. the stabilizer $W_\tau = \{w \in W \mid \tau^w = \tau\}$ is trivial. The intertwining operator $\mathcal{A}_w^\tau : I(\tau) \rightarrow I(\tau^w)$ is defined by

$$\mathcal{A}_w^\tau(f)(g) := \int_{N_w} f(wng)dn$$

where $N_w := N \cap w^{-1}N_-w$, with N_- being the unipotent radical of opposite parabolic P_- which corresponds to the negative roots Δ^- . The integral is convergent when $|e_\alpha(\tau)| < 1$ for all $\alpha \in \Delta^+$ such that $w\alpha \in \Delta^-$ (cf. [Bum] Proposition 63), and may be meromorphically continued to all τ . It has the property that for $x, y \in W$ with $\ell(xy) = \ell(x) + \ell(y)$, then

$$\mathcal{A}_y^{\tau^x} \mathcal{A}_x^\tau = \mathcal{A}_{xy}^\tau.$$

The Casselman's basis $\{f_w^\tau\}_{w \in W}$ of $I(\tau)^B$ is defined as follows. $f_w^\tau \in I(\tau)^B$ and

$$\mathcal{A}_y^\tau f_w^\tau(1) = \begin{cases} 1 & \text{if } y = w \\ 0 & \text{if } y \neq w. \end{cases}$$

M.Reeder characterizes this using the action of affine Hecke algebra (cf. [Re2] Section 2). The affine Hecke algebra $\mathcal{H} = \mathcal{H}(G, B)$ is the convolution algebra of B bi-invariant locally constant functions on G with values in \mathbb{C} . By the theorem of Iwahori-Matsumoto it can be described by generators and relations. The basis $\{T_w\}_{w \in \widetilde{W}_{aff}}$ consists of characteristic functions $T_w := ch_{BwB}$ of double coset BwB . Let \mathcal{H}_W be the Hecke algebra of the finite Weyl group W generated by the simple reflections s_α for simple roots $\alpha \in \Sigma$. As a vector space \mathcal{H} is the tensor product of two subalgebras $\mathcal{H} = \Theta \otimes \mathcal{H}_W$. The subalgebra Θ is commutative and isomorphic to the coordinate ring of the complex torus T with a basis $\{\theta_a \mid a \in A/A_0\}$, where θ_a is defined as follows (cf. [Lu2]). Define $A^- := \{a \in A \mid |\alpha(a)|_F \leq 1 \ \forall \alpha \in \Sigma\}$. For $a \in A$, choose $a_1, a_2 \in A^-$ such that $a = a_1 a_2^{-1}$. Then $\theta_a = q^{(\ell(a_1) - \ell(a_2))/2} T_{a_1} T_{a_2}^{-1}$ where for $x \in G$, $\ell(x)$ is the length function defined by $q^{\ell(x)} = [BxB : B]$ and $T_x \in \mathcal{H}$ is the characteristic function of BxB .

By Lemma (4.1) of [Re1], there exists a unique $f_w^\tau \in I(\tau)_w \cap I(\tau)^B$ for each $w \in W$ such that

- (1) $f_w^\tau(w) = 1$ and
- (2) $\pi(\theta_a)f_w^\tau = \tau^w(a)f_w^\tau$ for all $a \in A$.

Here $I(\tau)_w := \{f \in I(\tau) \mid \text{support of } f \text{ is contained in } \bigcup_{x \geq w} PxP\}$.

4.3 Transition coefficients

Let

$$f_w^\tau = \sum_{w \leq v} a_{w,v}(\tau) \varphi_v^\tau$$

and

$$\varphi_w^\tau = \sum_{w \leq v} b_{w,v}(\tau) f_v^\tau.$$

The Casselman's problem is to find an explicit formula for $a_{w,v}(\tau)$ and $b_{w,v}(\tau)$.

To relate the results in Sections 2 and 3 with the Casselman's problem, in this subsection we specialize the parameters $t_1 = -q^{-1}$, $t_2 = 1$ and take tensor product with the complex field \mathbb{C} . For example, the Yang-Baxter basis Y_w will become a $Q_{t_1, t_2}(\Lambda) \otimes \mathbb{C}$ basis in $H_{t_1, t_2}^{Q(\Lambda)}(W)_{\mathbb{C}} = H_{t_1, t_2}^{Q(\Lambda)}(W) \otimes \mathbb{C}$. The generic Demazure-Lusztig operator defined in Section 3 will become

$$y_i := A_i \delta_i + B_i \text{ where } A_i := \frac{-q^{-1} + e^{-\alpha_i}}{1 - e^{\alpha_i}}, B_i := \frac{-q^{-1} + 1}{1 - e^{-\alpha_i}}.$$

Then $(y_i + q^{-1})(y_i - 1) = 0$.

Theorem 3. *We identify e^α with e_α (cf. Remark 4). Then,*

$$a_{w,v}(\tau) = \tilde{p}(w, v)(\tau)|_{t_1 = -q^{-1}, t_2 = 1}$$

$$b_{w,v}(\tau) = p(w, v)(\tau)|_{t_1 = -q^{-1}, t_2 = 1}.$$

Proof. $b_{w,v}$'s satisfy the same recurrence relation (Proposition 5 with $t_1 = -q^{-1}$, $t_2 = 1$) as $p(w, v)$'s (cf. [Re2] Proposition (2.2)). The initial condition $b_{w,w} = p(w, w) = 1$ leads to the second equation. The first equation then also holds. Note that the $b_{y,w}$ in [Re2] is our $b_{w,y}$. □

Remark 6. *There is also a direct proof that does not use recurrence relation cf. [NN].*

Corollary 3. *We have a closed formula for $a_{w,v}(\tau)$ and $b_{w,v}(\tau)$ by Corollary 2 and Theorem 1.*

Corollary 4. *For $v \in W$, we have*

$$\sum_{w \leq v} b_{w,v} = \prod_{\beta \in R(v)} \frac{1 - q^{-1}e^\beta}{1 - e^\beta},$$

and

$$\sum_{w \leq v} b_{w,v} (-q^{-1})^{\ell(w)} = \prod_{\beta \in R(v)} \frac{1 - q^{-1}}{1 - e^\beta}.$$

Proof. When $t_1 = -q^{-1}, t_2 = 1$, we can specialize h_i to 1 and we get the first equation from the definition of Y_v , since $1 + \frac{(1-q^{-1})e^\beta}{1-e^\beta} = \frac{1-q^{-1}e^\beta}{1-e^\beta}$. We can also specialize h_i to $-q^{-1}$ and $-q^{-1} + \frac{(1-q^{-1})e^\beta}{1-e^\beta} = \frac{1-q^{-1}}{1-e^\beta}$ gives the second equation. \square

Remark 7. *The left hand side of the first equation in Corollary 4 is $m(e, v^{-1})$ in [BN]. So this gives another proof of Theorem 1.4 in [BN].*

4.4 Whittaker function

M.Reeder [Re2] specified a formula for the Whittaker function $\mathcal{W}_\tau(f_w^\tau)$ and using $b_{w,v}$, he got a formula for $\mathcal{W}_\tau(\varphi_w^\tau)$. For $a \in A$, let $\lambda_a \in X^*(T)$ be

$$\lambda_a(z \otimes \mu) = z^{val(\mu(a))} \text{ for } z \in \mathbb{C}^*, \mu \in X^*(A).$$

Formally the result of M.Reeder [Re2] Corollary (3.2) is written as follows. For $w \in W$ and $a \in A^-$,

$$\mathcal{W}(\varphi_w)(a) = \delta^{1/2}(a) \sum_{w \leq y} b_{w,y} y \left[\lambda_a \prod_{\beta \in R^+ - R(y)} \frac{1 - q^{-1}e^\beta}{1 - e^{-\beta}} \right] \in \mathbb{C}[T].$$

Then using Corollary 3, we have an explicit formula of $\mathcal{W}(\varphi_w)(a)$.

4.5 Relation with Bump-Nakasuji's work

Now we explain the relation between this paper and Bump-Nakasuji [BN]. First of all, the notational conventions are slightly different. Especially in the published [BN] the natural base and intertwiner are differently parametrized. The natural basis ϕ_w in [BN] is our $\varphi_{w^{-1}}$. The intertwiner M_w in [BN] is our $\mathcal{A}_{w^{-1}}$ so that if $\ell(w_1 w_2) = \ell(w_1) + \ell(w_2)$, $M_{w_1 w_2} = M_{w_1} \circ M_{w_2}$ while $\mathcal{A}_{w_1 w_2} = \mathcal{A}_{w_2} \mathcal{A}_{w_1}$.

In the paper [BN], another basis $\{\psi_w\}_{w \in W}$ for the space $I(\tau)^B$ was defined and compared with the Casselman's basis. They defined $\psi_w := \sum_{v \geq w} \varphi_v$ and expand this as $\psi_w = \sum_{v \geq w} m(w, v) f_v$ and conversely $f_w = \sum_{v \geq w} \tilde{m}(w, v) \psi_v$. They observed that the transition coefficients $m(w, v)$ and $\tilde{m}(w, v)$ factor under certain condition. Let $S(w, v) := \{\alpha \in R^+ | w \leq s_\alpha v < v\}$ and $S'(w, v) := \{\alpha \in R^+ | w < s_\alpha w \leq v\}$. Then the statements of the conjectures are as follows.

Conjecture 1. ([BN] Conjecture 1.2) Assume that the root system R is simply-laced. Suppose $w \leq v$ and $|S(w, v)| = \ell(v) - \ell(w)$, then

$$m(w, v) = \prod_{\alpha \in S(w, v)} \frac{1 - q^{-1}z^\alpha}{1 - z^\alpha}.$$

Conjecture 2. ([BN] Conjecture 1.3) Assume that the root system R is simply-laced. Suppose $w \leq v$ and $|S'(w, v)| = \ell(v) - \ell(w)$, then

$$\tilde{m}(w, v) = (-1)^{\ell(v) - \ell(w)} \prod_{\alpha \in S'(w, v)} \frac{1 - q^{-1}z^\alpha}{1 - z^\alpha}.$$

Proposition 10. Conjecture 1.2 and Conjecture 1.3 in [BN] are equivalent.

Proof. We can show $m(w, v) = \sum_{w \leq z \leq v} p(z, v)$ and $\tilde{m}(w, v) = \sum_{w \leq z \leq v} (-1)^{\ell(v) - \ell(z)} \tilde{p}(w, z)$.

Then it follows by the Theorem 1 that $\tilde{m}(w, v) = (-1)^{\ell(v) - \ell(w)} m(vw_0, ww_0)$. As $S'(w, v) = S(vw_0, ww_0)$ we get the desired conclusion. \square

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